Indian Statistical Institute
End-Semestral Examination
Algebra I
November-2017

Max Marks: 100 Time: 3 hours.

Answer question 1 and any **five** from the rest.

1. Prove or disprove.
   (a) A group of order 35 is cyclic.
   (b) There exists exactly 90 elements of order 7 in a simple group of order 105.
   (c) If $H \trianglelefteq G$, such that H intersects the commutator subgroup of G trivially, then $H \subseteq Z(G)$.
   (d) The automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\mathbb{Z}_6$.
   (e) S_4 and D_{24} are isomorphic. (6×5)

2. (a) Show that the commutator subgroup of a group G is a normal subgroup of G.
   (b) Show that the commutator subgroup of S_n is A_n, for all $n \geq 3$. (4+10)

3. (a) Let G be a group acting on a set X and let $x \in X$. Let $\mathrm{Orb}(x)$ denote the orbit of x and G_x denote the stabiliser of x. Show that $|\mathrm{Orb}(x)| = |G : G_x|$.
   (b) Let p be a prime, $o(G) = p^n$ for some $n \geq 1$ and let G act on a finite set X. Let $X_0 = \{x \in X \text{ such that } g.x = x \ \forall g \in G\}$ be the fixed point set. Show that $|X| \equiv |X_0| \bmod p$. (7+7)

4. (a) State Sylow's theorems.
   (b) Let $o(G) = pqr$ where p, q, r are primes with $p < q < r$. Show that G has a normal Sylow subgroup for either p, q or r. (6+8)

5. Determine the Sylow subgroups of A_5. (14)

6. Let G be a finite abelian group. Show that G is the (internal) direct product of its Sylow subgroups. (14)

7. (a) Define (external) semidirect product of two groups H and K.
   (b) Classify all groups of order 12 where the Sylow 3-subgroup is normal. (4+10)